Computational Models - Lecture 6[1]

Handout Mode

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April 4/6, 2016

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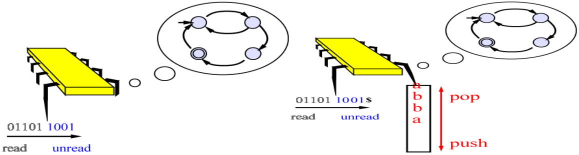
[1] Based on frames by Benny Chor, Tel Aviv University, modifying frames by Maurice Herlihy, Brown University. Also with modifications of Yishay Mansour.

Outline

- Push Down Automata (PDA)
- Closure properties for CFL and testing properties.
- Equivalence of CFGs and PDAs



- Sipser's book, 2.1, 2.2 & 2.3

Part I

Push-Down Automata: Review

Diagram Notation

When drawing the automata diagram, we use the following notation

- Transition from state q to state q' labelled by $a, b \to c$ means $(q', c) \in \delta(q, a, b)$,
  and informally means the automata
  - read a from input
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- Meaning of ε transitions ((informally):
    - $a = \varepsilon$: don't read input
    - $b = \varepsilon$: don't pop any symbol
    - $c = \varepsilon$: don't push any symbol

How to define $\widehat{\delta}(q, w, s)$ contains (q', s')?

Given (start) state q, substring w of the input, and s, s' descriptions of strings on a stack:
There is a legal way to get from state q with stack contents s to state q' with stack contents s' by reading from w at each step.

Model of Computation

The following is with respect to $M = (Q, \Sigma, \Gamma, \delta, q_0, F)$.

Definition 1 (δ^*)

For $w \in \Sigma^*$ let $\widehat{\delta}(q, w, s)$ be all pairs $(q', s') \in Q \times \Gamma^*$ for which exist $w'_1, \ldots, w'_m \in \Sigma_\varepsilon$, states $r_1, \ldots, r_m \in Q$ and strings $s_0, s_1, \ldots s_m \in \Gamma^*$ s.t.:

1. $w = w'_1, \ldots, w'_m$, $r_0 = q$, $r_m = q'$, $s_0 = s$ and $s_m = s'$
2. For every $i \in \{0, \ldots, m-1\}$ exist $a, b \in \Gamma_\varepsilon$ and $t \in \Gamma^*$ s.t.:
   - **2.1** $(r_{i+1}, b) \in \delta(r_i, w'_{i+1}, a)$
   - **2.2** $s_i = at$ and $s_{i+1} = bt$

Namely, $(q', s') \in \widehat{\delta}(q_0, w, \varepsilon)$ if after reading w (possibly with in-between ε moves), M can find itself in state q' and stack value s'.

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Namely, $(q', s') \in \widehat{\delta}(q_0, w, \varepsilon)$ if after reading w (possibly with in-between ε moves), M can find itself in state q' and stack value s'.

- M accepts $w \in \Sigma^*$ if $\exists q' \in \mathcal{F}$ such that $(q', t) \in \widehat{\delta}(q_0, w, \varepsilon)$ for some t.

Knowing when stack is empty

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Solution

1. Start by pushing $\$$ onto stack.
2. When you see it again, stack is empty.

Example 3 – Palindrome

A palindrome is a string w satisfying $w = w^{\mathcal{R}}$.

- “Madam I’m Adam”
- “Dennis and Edna sinned”
- “Red rum, sir, is murder”
- “Able was I ere I saw Elba”
- “In girum imus nocte et consumimur igni” (Latin: "we go into the circle by night, we are consumed by fire”.)
- “νιψον ανομηματα μη μοναν οψιν”
- Palindromes also appear in nature. For example as DNA restriction sites – short genomic strings over $\{A, C, T, G\}$, being cut by (naturally occurring) restriction enzymes.

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What the difference from $\{ww^R\}$?

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Algorithm 2

Input: $x \in \Sigma^*$

1. Start pushing x into stack.
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4. `Accept` If end of input occurs together with emptying of stack.

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- This PDA accepts palindromes of even length over the alphabet (all lengths is an easy modification).
- Again, non-determinism (at which point to make the switch) seems necessary.

PDA Languages

The Push-Down Automata Languages, $\mathcal{L}_{\text{PDA}}$, is the set of all languages that can be described by some PDA:

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- $\mathcal{L}_{\text{PDA}} = \mathcal{L}_{\text{CFG}}$!!!

Part II

Closure Properties

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    - Star:$S_{new} \rightarrow \varepsilon \mid S_{old} \mid S_{old} S_{new}$
- What about complement and intersection?

Intersection idea?

- Idea: Can't we run two PDA's in parallel, and accept iff both accept??

Intersection

$$\begin{array}{ll} S_1 \to A_1 B_1 & \quad S_2 \to A_2 B_2 \\ A_1 \to 0A_1 1|\varepsilon & \quad A_2 \to 0A_2|\varepsilon \\ B_1 \to 2B_1|\varepsilon & \quad B_2 \to 1B_2 2|\varepsilon \end{array}$$

$$\mathcal{L}_1 = 0^n 1^n 2^* \qquad \mathcal{L}_2 = 0^* 1^n 2^n$$

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- So.. we can't we run two PDA's in parallel, and accept iff both accept.
- What about intersection of a CFL with a regular language?

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  - Formal details omitted (but you should be able to figure them out).

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  - So $\mathcal{L}$ is not a context free language
- This could also be established using pumping lemma, but proof above is more elegant.

Closure under complementation?

CFLs are closed under union. If CFLs are also closed under complementation, then they would be closed under intersection because of:

$$\mathcal{L}_1 \cap \mathcal{L}_2 = \overline{\overline{\mathcal{L}_1} \cup \overline{\mathcal{L}_2}}$$

But, CFLs are not closed under intersection, so they cannot be closed under complementation.

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- Let's restate:
  - $\mathcal{L}$ are the strings of length 2ℓ for which for all $1 \leq i \leq \ell$, $w_i = w_{i+\ell}$.
  - $\overline{\mathcal{L}}$ are strings for which either (1) $|w|$ is odd, or (2) $|w|$ is even and there exists i, for which $w_i \neq w_{i+\ell}$.

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- CFG for $\overline{\mathcal{L}}^{0}_{even}$
  $S \rightarrow AB$
  $A \rightarrow CAC \mid 0$
  $B \rightarrow CBC \mid 1$
  $C \rightarrow 0 \mid 1$

A PDA for $\overline{\mathcal{L}}^{0}_{even}$

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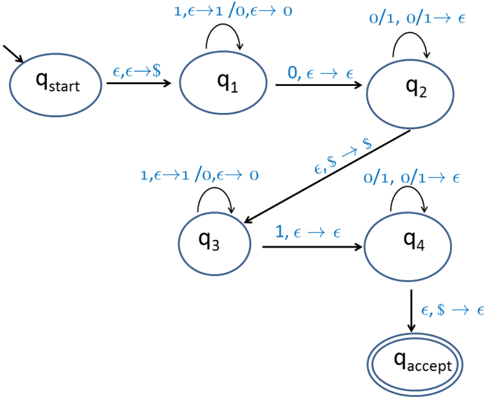
Idea: Guess $k, j \geq 0$, and accept w if it is of the form:
$\{0,1\}^k 0 \{0,1\}^k \{0,1\}^j 1 \{0,1\}^j$



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Part III

Algorithmic Questions

Emptiness of CFGs

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- Better Idea: Can the start variable generate a string of terminals?
- A more holistic approach: Can a particular variable generate a string of terminals?

Checking Emptiness

Idea: Mark variables that can produce a string of terminals

1. Mark all terminal symbols in G.
2. Repeat until no new variable become marked:

   Mark any A where $A \to U_1 U_2 \dots U_k$ and all U_i have already been marked.
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Correctness?

Recall cleanup in the CNF conversion process?

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Question 5

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- We are not prepared to prove this remarkable fact (yet).

Finiteness of CFGs

Question 7

Given a CFG G, is $|\mathcal{L}(G)|$ finite?

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First, a useful subroutine.

CLEANUP: Removing redundant variables and terminals

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2. Repeat until no new variable become marked:

   Mark any A where $A \to U_1 U_2 \ldots U_k$ and all U_i have already been marked.
3. Remove all unmarked variables, and any rule they appear in.
4. If S is removed, then $\mathcal{L}(G) = \emptyset$.
5. Remove any variable A not reachable from S.
6. Remove any terminal which does not appear in some rule.

Back to finiteness of CFGs

Question 8

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Correctness?

CFGs Inherent Ambiguity

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Given a CFG G, is $\mathcal{L}(G)$ inherently ambiguous?

This means that for any CFG that generates $\mathcal{L}(G)$, there is a word in the language with two different parse trees.

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Fact 10

There is no algorithm to solve CFG inherent ambiguity.

- We will not prove this fact, yet you want to know it to put things in context.

When Are Two CFGs equivalent?

Question 11

Given two CFG G_1 and G_2, test if $L(G_1) = L(G_2)$.
Is there an algorithm to solve this problem?

Part IV

Equivalence Theorem

The CFG–PDA Equivalence Theorem

Theorem 12

$\mathcal{L}_{\text{PDA}} = \mathcal{L}_{\text{CFG}}$ *: A language is context free if and only if some pushdown automata accepts it.*

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Proof sketch follows.

CFL $\Longrightarrow$ PDA

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$\mathcal{L}_{\text{CFG}} \subseteq \mathcal{L}_{\text{PDA}}$ *: If a language is context free, then some PDA accepts it.*

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- Let $\mathcal{L}$ be a context-free language, and let $G = (V, \Sigma, R, S)$ be context-free grammar for $\mathcal{L}$
- We build a PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, F)$, such that on input w it "figures out" if there is a derivation of w using G.

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How does P figure out which substitution to make?

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Answer: It guesses.

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   Can we justify it?

2. When deriving a word from a CFL, we always substitute the left most variable

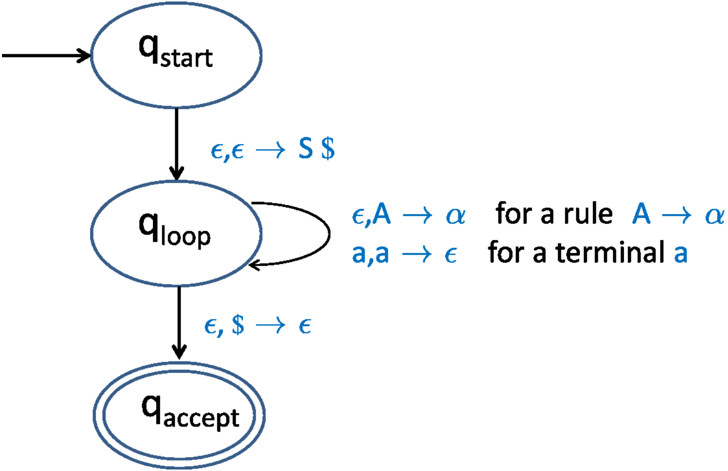
   Does it change the derived language?

Informal Description of P

Algorithm 15 (P)

```
1. Push $S\$$ on stack
2. While top of the stack $t$ is not $\$$:
    2.1 If $t$ is variable $A$,
        (non-deterministically) select rule $A \to \alpha$ and substitute (i.e. push $\alpha$ to stack).
    2.2 If $t$ is a terminal $a$,
        read next input and compare; Reject if different.
    2.3 Accept if end of input and stack is empty
```

State Diagram for *P*

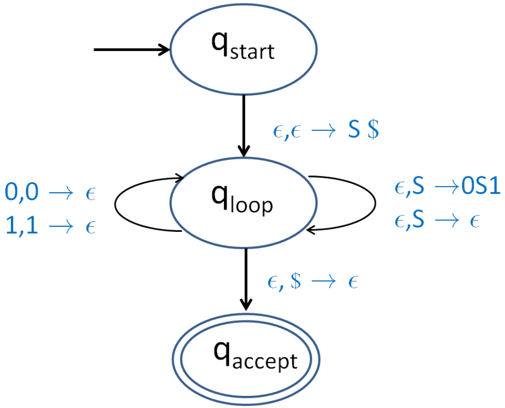


Example

consider the CFG:

$S \rightarrow 0S1|\varepsilon$.

The related PDA:



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$S \xrightarrow{*} \alpha$ iff $\alpha = \alpha_1\alpha_2$ such that $(q_{loop}, \alpha_2\$) \in \widehat{\delta}(q_{loop}, \alpha_1, \$)$.

(note that α_1 is made of terminals, α_2 can be variables and terminals)

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Does the above yield that $\mathcal{L}(P) = \mathcal{L}(G)$?

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- It is easy to see that $(q_{loop}, tw_2\$) \in \widehat{\delta}(q_{loop}, \alpha'_1 w_1, S\$)$.
- To complete the proof take $\alpha_1 = \alpha'_1 w_1$ and $\alpha_2 = tw_2$.

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- Note that $(q_{loop}, \alpha_2'\$) \in \widehat{\delta}(q_{loop}, \alpha_1', \$)$.

$\alpha = \alpha_1\alpha_2$ such that $(q_{loop}, \alpha_2\$) \in \widehat{\delta}(q_{loop}, \alpha_1, \$) \implies S \xrightarrow{*} \alpha$

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PDA $\implies$ CFL

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$\mathcal{L}_{\text{PDA}} \subseteq \mathcal{L}_{\text{CFG}}$: *If a PDA accepts a language then it is context free.*

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Can we justify the above?

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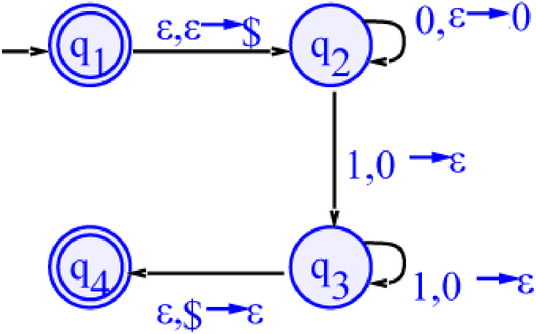
Example PDA to CFG



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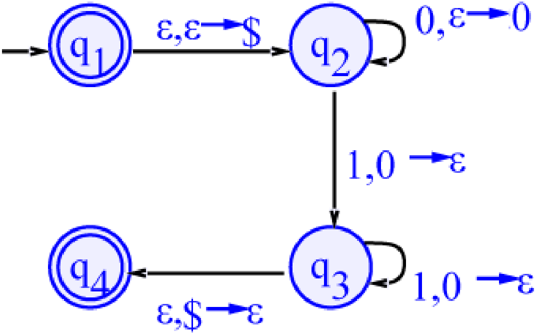

$A_{q_1,q_4} \to A_{q_2,q_3}$
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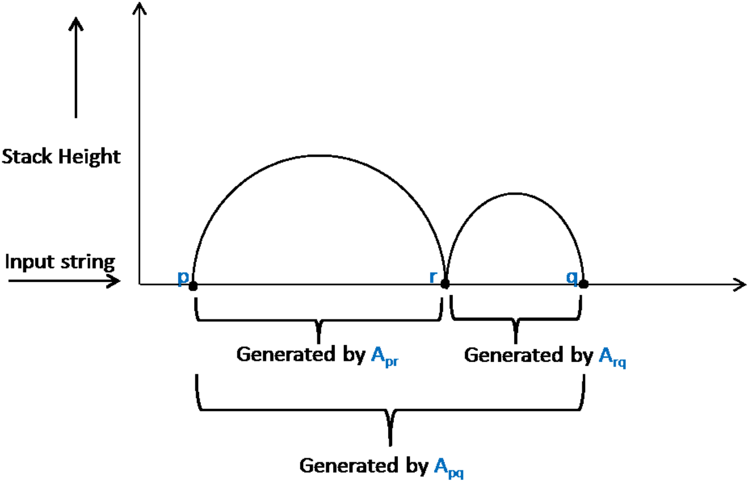
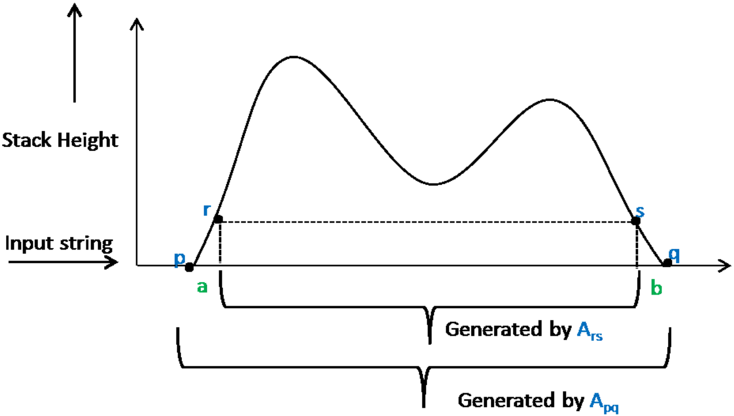
$A_{q_2,q_3} \to 0A_{q_2,q_2}1$.
$A_{q_2,q_2} \to \varepsilon$.

PDA Computation corresponding to $A_{pq} \rightarrow A_{p,r}A_{r,q}$



PDA Computation corresponding to $A_{pq} \rightarrow aA_{r,s}b$



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Proof by induction on the number of derivation rules/ transitions

A Short Summary

- Regular Languages ≡ Finite Automata.
- Context Free Languages ≡ Push Down Automata.
- Closure properties of regular languages and of CFLs.
- Most algorithmic problems for finite automata are solvable.
- Some algorithmic problems for finite automata are not solvable.
- Pumping lemmata for both classes of languages.
- There are additional languages out there.

The View Over The Horizon