Computational Models — Lecture 12[1]

Handout Mode

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[1] Based on frames by Benny Chor, Tel Aviv University, modifying frames by Maurice Herlihy, Brown University. Also with modifications by Yishay Mansour.

Talk Outline

- Reminder – deterministic and nondeterministic time classes
- $\mathcal{NP}$ and verifiability
- Additional NP languages
- The class *co*-$\mathcal{NP}$
- $\mathcal{P}$ Verses $\mathcal{NP}$
- NP-completeness
- Satisfiability
- Reductions

- Sipser's book, 7.4–7.5

Reminder – Deterministic Time

Definition 1 (deterministic time)

A deterministic TM M runs in time t, where $t : \mathbb{N} \mapsto \mathbb{N}$, if for every input w, the number of steps that $M(w)$ uses is at most $t(|w|)$.

Definition 2 (DTIME)

For $t : \mathbb{N} \mapsto \mathbb{N}$, let
$\mathrm{DTIME}(t) = \{L \subseteq \Sigma^* : \ L \text{ is decided by an } \ O(t)\text{-time single tape TM}\}$

The bound on the running time is required to hold also for input not in the language (i.e., L).

Definition 3 ($\mathcal{P}$)

$\mathcal{P} = \bigcup_{c \geq 0} \mathrm{DTIME}(n^c)$

Runtime on inputs of size n is $O(n^k)$ for k fixed constant (k can be $1, ..., 3010, ...$ but not $\log n, \log \log n$)

Reminder – Non-Deterministic Time

Definition 4 (nondeterministic time)

A non-deterministic TM N runs in time t, where $t: \mathbb{N} \mapsto \mathbb{N}$, if for every input w, the maximum number of steps that $N(w)$ uses on any branch of its computation tree, is at most $t(|w|)$.

Notice that also non-accepting branches must reject within the required time.

Definition 5 (NTIME)

For $t \mapsto \mathbb{N} \mapsto \mathbb{N}$ let $\mathrm{NTIME}(t) =$
$\{L \subseteq \Sigma^* : \ L$ is decided by an $O(t)$-time single tape NTM$\}$

Definition 6 ($\mathcal{NP}$)

$\mathcal{NP} = \bigcup_{c \geq 0} \mathrm{NTIME}(n^c)$

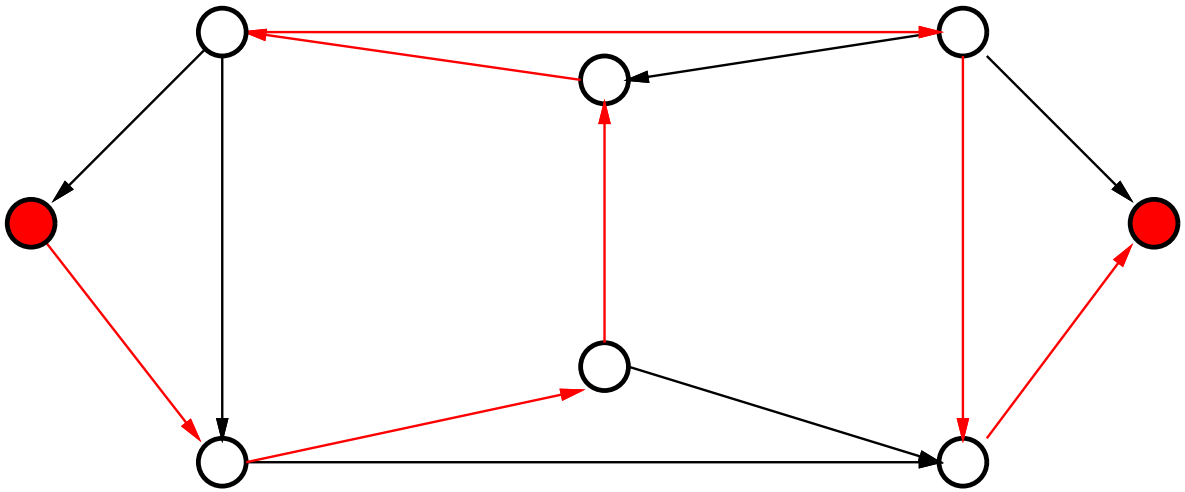
Are problems in $\mathcal{NP}$ solvable in polynomial time?

We don't know!

but what we do know is this:

there is a large class of fundamental problems that are "computationally equivalent",
i.e., if ONE is efficiently solvable then ALL are efficiently solvable

Hamiltonian Path



A Hamiltonian path in a directed G visits each node exactly once.

$$\text{HAMPATH} = \{\langle G, s, t\rangle : G \text{ has Hamiltonian path from } s \text{ to } t\}$$

Question 7

How hard is it to decide HAMPATH?

Easy to obtain exponential time algorithm:

- Generate each potential path
- Check whether it is Hamiltonian

HAMPATH $\in \mathcal{NP}$

Here is an NTM that decides HAMPATH in polynomial time.

Algorithm 8 (N)

On input $\langle G = (V, E), s, t \rangle$,

1. Guess a list of numbers $p_1, \ldots, p_m$, where $m = |V|$ and $1 \leq p_i \leq m$.
2. `Accept` if all the following hold (otherwise `Reject`):
   - No repetitions in list
   - $p_1 = s$ and $p_m = t$.
   - $(p_i, p_{i+1}) \in E$ for every $1 \leq i \leq m - 1$

- How does a TM guess a string?

Claim 9

N runs in polynomial time

Verifiability of HAMPATH

This problem has a very interesting feature: polynomial verifiability:

> *We don't know a fast way to find a Hamiltonian path, but we can check whether a given path is Hamiltonian in polynomial time.*

Verifying correctness of a given path is much easier than determining whether one exists

An efficiently verifiable proof

- What if for each $x \in L$, there is *always* additional information – a *string* c, which convinces us in *poly time* that $x \in L$?
  - We call such a c a certificate (also known as a *proof* or a *witness*)
  - But what does "convinces us" actually mean?
    - ⋆ e.g. Certificate for $\langle G, s, t \rangle \in$ HAMPATH is Hamiltonian path from s to t.
      Easy to verify in time polynomial in $|\langle G, s, t \rangle|$ whether given path is Hamiltonian.
- What about $x \notin L$? it better be that no such string exists!

A proof that convinces us exists when $x \in L$, but not when $x \notin L$!

Verifiability

Definition 10 (verifier)

Algorithm V is a verifier for a language L, if

- $x \in L \implies \exists c \in \{0,1\}^*$ s.t. $V(x,c) = 1$.
- $x \notin L \implies \nexists c \in \Sigma^*$ s.t. $V(x,c) = 1$.

- If V accepts (x,c) (i.e., outputs 1), the string c is called a certificate (proof/witness)
  - Note: we might not know how to find c efficiently
- A polynomial verifier runs in polynomial time in $|x|$ (i.e., in the length of its left-hand-side input parameter).
  - A language L is polynomially verifiable, if it has a polynomial verifier.
  - Not all languages are known to be polynomially verifiable.

NP and Verifiability

Theorem 11

A language is in $\mathcal{NP}$ iff it has a polynomial time verifier.

$\mathcal{NP} \implies$ Verifiability

Claim 12

If L is decided by a polynomial-time NTM N, then L has a poly-time verifier.

Idea: guesses of the NTM are the certificate

Proof: Assume for simplicity that at each step of N, the number of possible non-deterministic moves is at most two.

Algorithm 13 (V)

On input (x, c):

1. Emulate $N(x)$, treating each symbol of c as a description of the non-deterministic choice in each step of N.
2. `Accept` if this branch accepts; Otherwise `Reject`.

♣

Without the simplifying assumption?

Verifiability $\Longrightarrow \mathcal{NP}$

Claim 14

If L has a poly-time verifier, then it is decided by some polynomial-time NTM.

Idea: guess c and then use V to verify it

Proof: Let V be poly-time verifier for L of running time $p(n)$ for some $p \in$ poly.

Algorithm 15 (N)

On input $x \in \{0,1\}^n$:

1. Guess a string c of length $p(n)$.
2. Emulate V on $\langle x, c \rangle$
3. `Accept` if V accepts; Otherwise `Reject`.

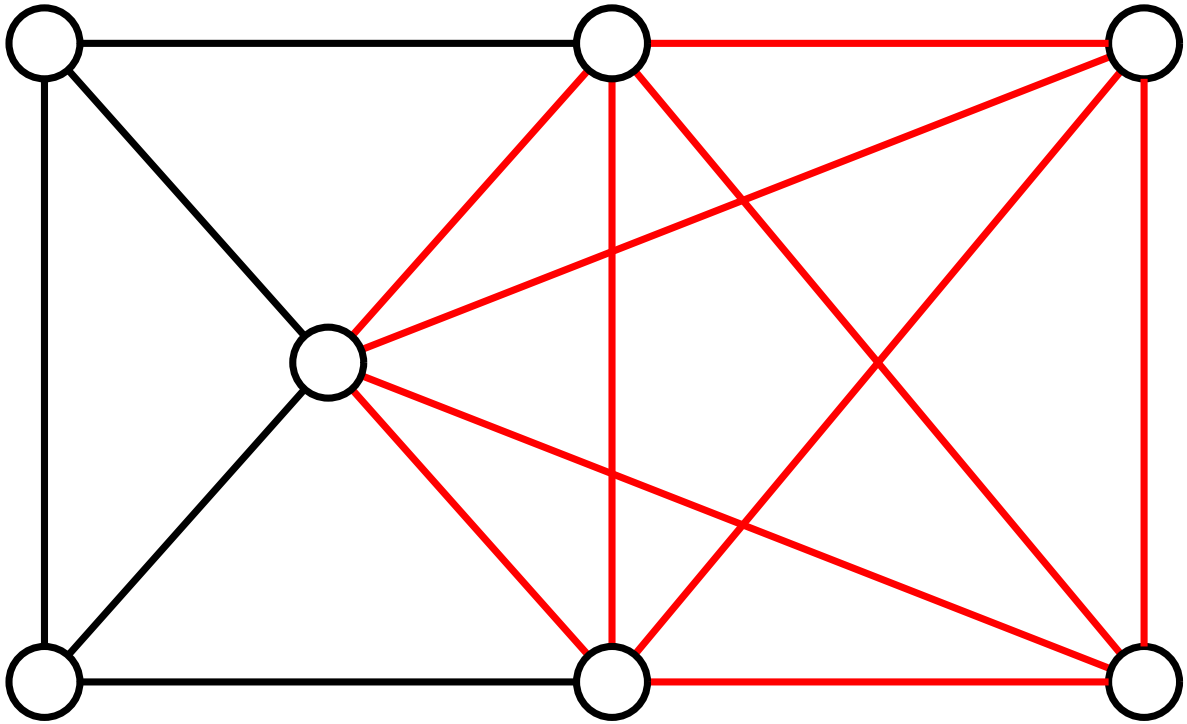
♣

- if $x \in L$ then there is some c such that $V(x,c) =$`Accept`, so N accepts!
- if $x \notin L$, then for all c, $V(x,c) =$`Reject`, so N rejects!
- Why does it suffice to guess a string of length $p(n)$?

Section 1

A few more NP languages

CLIQUE



A clique in a graph is a subgraph where every two nodes are connected by an edge.

A k-clique is a clique of size k.

Question 16

What is the largest k-clique in the figure?

CLIQUE cont.

$$\text{CLIQUE} = \{\langle G, k\rangle : G \text{ is an undirected graph with a } k\text{-clique}\}$$

Theorem 17

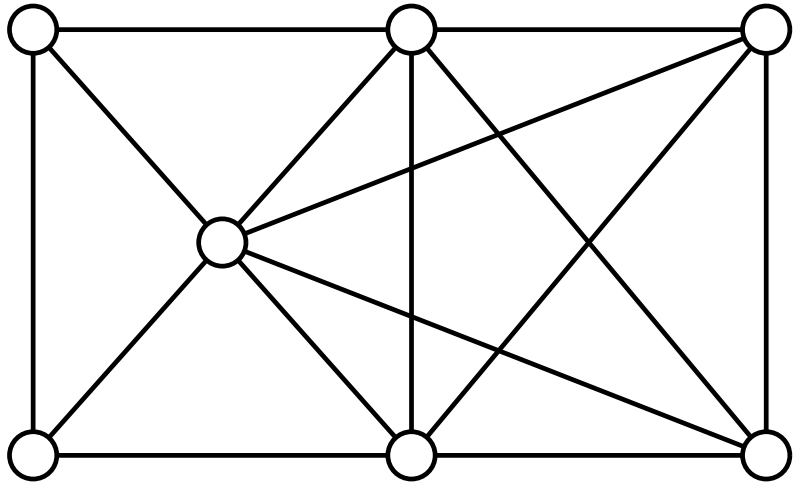
CLIQUE $\in \mathcal{NP}$

Proof's idea: The clique is the certificate.

Algorithm 18 (V)

On input $(\langle G, k\rangle, c)$
`Accept` if c is a k-clique subgraph of G;
Otherwise `Reject`.

Independent Set



An independent set in a graph is a set of vertexes, no two of which are linked by an edge.

A k-IS is an independent set of size k.

Question 19

What is the largest k-IS in the figure?

IND-SET cont.

$$\text{IND-SET} = \{\langle G, k\rangle : G \text{ contains an independent set of size } k\}$$

Theorem 20

IND-SET $\in \mathcal{NP}$

Proof's idea: The independent set is the certificate.

Algorithm 21 (V)

On input $(\langle G, k\rangle, c)$
`Accept` if c is a k-IS of G (no edges between nodes in c, and $|c| = k$);
Otherwise `Reject`.

Section 2

The Class coNP

The Class *co-*$\mathcal{NP}$

$\overline{\text{CLIQUE}} = \{\langle G, k\rangle : G$ is an undirected graph with no k-clique$\}$ seems not to be member of $\mathcal{NP}$.

It seems harder to efficiently verify that something does not exist than to efficiently verify that something does exist.

Definition 22 (*co-*$\mathcal{NP}$)

co-$\mathcal{NP} = \{\mathsf{L} : \overline{\mathsf{L}} \in \mathcal{NP}\}$.

But.. we are not sure...So far, no one knows if *co-*$\mathcal{NP}$ is distinct from $\mathcal{NP}$.

Claim 23

$\mathcal{P} \subseteq$ *co-*$\mathcal{NP}$.

Proof?

$$\mathsf{L} \in \mathcal{P} \implies \overline{\mathsf{L}} \in \mathcal{P} \implies \overline{\mathsf{L}} \in \mathcal{NP} \implies \mathsf{L} \in \textit{co-}\mathcal{NP}$$

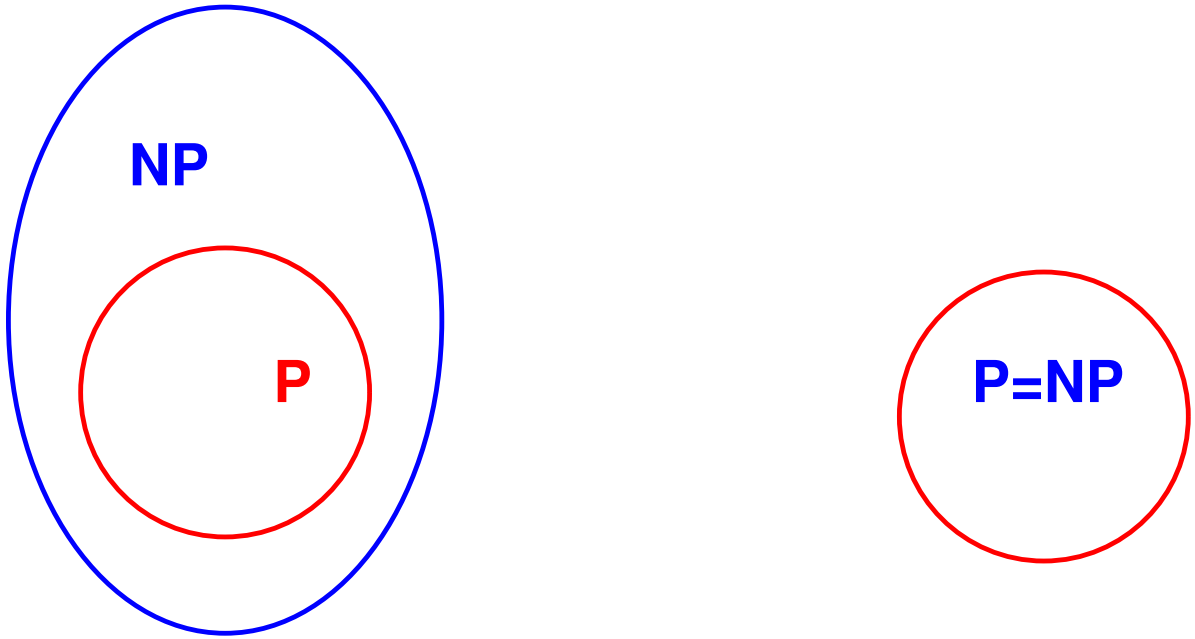
Be very careful!

Is Primality in $\mathcal{NP}$? *co-*$\mathcal{NP}$?
How would you prove that a number is prime without trying all divisors?
Actually it is in P! (not obvious at all)

Section 3

P vs. NP

$\mathcal{P}$ vs. $\mathcal{NP}$



The question $\mathcal{P} \stackrel{?}{=} \mathcal{NP}$ is one of the great unsolved mysteries in contemporary mathematics.

$\mathcal{P}$ vs. $\mathcal{NP}$

- Most computer scientists believe the two classes are not equal
- Most bogus proofs show them equal (?)
- One of 7 Clay Millenium Prize problems (1 million dollars)
- “Computer Science’s greatest intellectual export” (Papadimitriou 2007)

$\mathcal{P}$ Vs. $\mathcal{NP}$, cont.

If $\mathcal{P}$ differs from $\mathcal{NP}$, then the distinction between $\mathcal{P}$ and $\mathcal{NP} \setminus \mathcal{P}$ is meaningful and important.

- languages in $\mathcal{P}$ are tractable
- languages in $\mathcal{NP} \setminus \mathcal{P}$ are intractable

Until we can prove that $\mathcal{P} \neq \mathcal{NP}$, there is no hope of proving that a specific language lies in $\mathcal{NP} \setminus \mathcal{P}$.

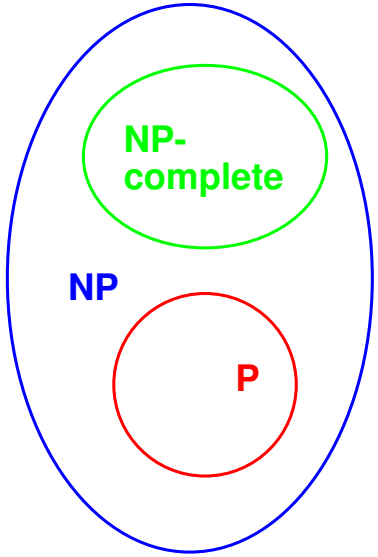
Nevertheless, we can prove statements of the form

If $\mathsf{A} \in \mathcal{NP} \setminus \mathcal{P}$, *then* $\mathsf{B} \in \mathcal{NP} \setminus \mathcal{P}$.

Section 4

NP Completeness

NP Completeness



The class of NP-complete languages are

- "hardest" languages in $\mathcal{NP}$
- If any NP-complete L $\in P$, then $\mathcal{NP} = \mathcal{P}$.

Question 24

Are there NP-complete languages?

Polynomial-Time Computable Functions

Definition 25 (poly-time computable functions)

A function $f : \Sigma^* \mapsto \Sigma^*$ is polynomial-time computable, if there is a poly-time deterministic TM that

- starts with input w, and
- halts with $f(w)$ on tape.

Polynomial-Time Reducibility

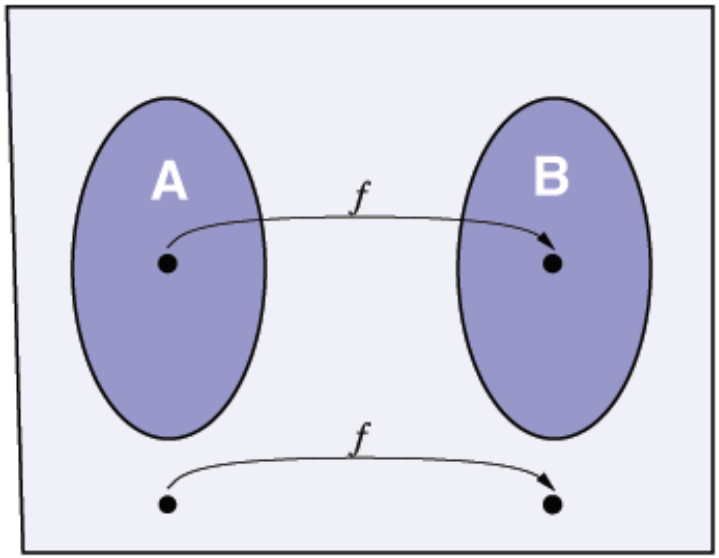
Definition 26

A language A is polynomial time mapping reducible to B, denoted $A \leq_P B$, if exists poly-time computable f such that

$$w \in A \iff f(w) \in B.$$

for every $w \in \Sigma^*$.
The function f is called a polynomial-time reduction from A to B.

The mapping f efficiently converts questions about membership in A to membership in B.



Reductions to $\mathcal{P}$

Theorem 27

If $A \leq_P B$ *and* $B \in \mathcal{P}$ *then* $A \in \mathcal{P}$.

Proof:

- Let f the reduction from A to B, computed by TM M_f.
  On input x, the TM M_f makes at most $c_f \cdot |x|^{a_f}$ steps.
- Let M_B be the poly-time decider for B.
  On input y, the TM M_B makes at most $c_B \cdot |y|^{a_B}$ steps.

Algorithm 28 (Decider M_A for A)

On input x, return $M_B(f(x))$

- M_A decides A
- Since $|f(x)| \leq c_f|x|^{a_f}$, running time of $M_B(x)$, is at most $c_B \cdot (c_f \cdot |x|^{a_f})^{a_B} = (c_B \cdot c_f^{a_B}) \cdot |x|^{a_f \cdot a_B} \in \text{poly}(|x|)$

  Hence, $A \in \mathcal{P}$

What $A \leq_P B$ tells us about B?

Question 29

Assume that $\{0^n1^n : n \geq 0\} \leq_P L$. Does it yield that $L \in \mathcal{P}$?

Answer: No. (Reduction in the wrong direction!)

Let $L = H_{TM}$ and define $f(x) = \begin{cases} M_{stop}, & x \in \{0^n1^n : n \in \mathbb{N}\} \\ M_{no-stop}, & \text{otherwise.} \end{cases}$.

$A \leq_P B$ does tell us that B is “at least as hard" as A.

NP Completeness, Formal Definition

Definition 30 ($\mathcal{NP}$-complete)

A language B is NP-complete, if

- B $\in \mathcal{NP}$, and
- Every A $\in \mathcal{NP}$ is poly-time reducible to B (i.e., A $\leq_P$ B)

We let $\mathcal{NPC}$ denote the class of all NP-complete languages.

Compare to

Definition 31 (RE-Complete)

A language B is RE-complete, if

- B $\in \mathcal{RE}$, and
- Every A $\in \mathcal{RE}$ is mapping reducible to B.

Why NP Completeness?

Theorem 32

If $\mathsf{B} \in \mathcal{NPC}$ *and* $\mathsf{B} \in \mathcal{P}$, *then* $\mathcal{P} = \mathcal{NP}$.

Proof: Immediately follows by Thm 27. ♣

To show $\mathcal{P} = \mathcal{NP}$ (and make an instant fortune, see www.claymath.org/millennium/P_vs_NP/), suffices to find a polynomial-time algorithm for any NP-complete problem.

Question 33

Is $\mathcal{NPC}$ empty?

$\mathcal{NPC}$ Is Not Empty

$A_{NP} = \{\langle M, x, 1^n\rangle : M \text{ is a TM} \wedge \exists c \in \Sigma^* \text{ s.t. } M(x, c) \text{ accepts within } n \text{ steps}\}$.

Theorem 34

$A_{NP} \in \mathcal{NPC}$

Proof:

- Clearly $A_{NP} \in \mathcal{NP}$.(?)
- Let $L \in \mathcal{NP}$, let V be a verifier for L and let $p \in poly$ be a bound on the running time of V (i.e., $V(x, \cdot)$ halts within $p(|x|)$ steps, for every $x \in \Sigma^*$)).
- Define $f(x) = \langle V, x, 1^{p(|x|)}\rangle$.
- Clearly f is poly-time computable and $x \in L \iff f(x) \in A_{NP}$.

♣

Finding Additional NP-complete Languages

Theorem 35

Assume that

1. $B \in \mathcal{NP}$
2. $A \in \mathcal{NPC}$ *and* $A \leq_P B$

then $B \in \mathcal{NPC}$.

Proof: Home exercise . . . ♣

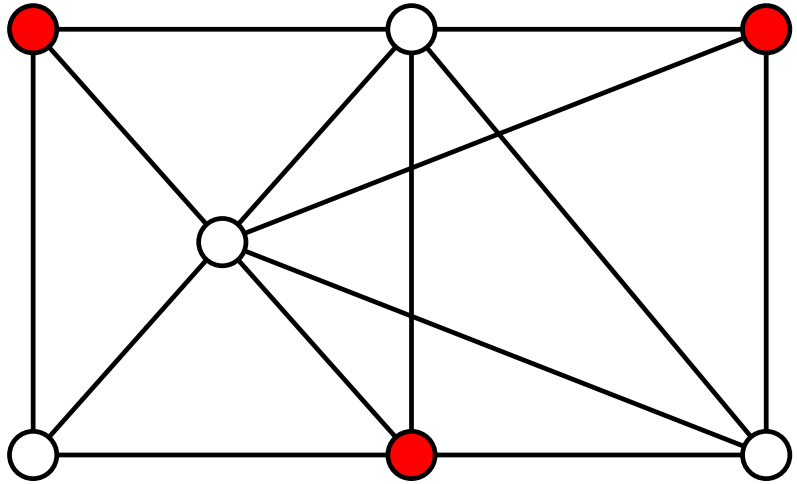
We would like to find $L \in \mathcal{NPC}$ that is “natural" and “easy" to reduce to.

The most useful such language is the language of satisfied formulas. But, we will start with clique.

Section 5

A first example of an NP-completeness reduction: Independent Set

Independent Set



Definition 36

An independent set in a graph is a set of vertexes, no two of which are linked by an edge.

The language of independent sets:

$$\text{IND-SET} = \{\langle G, k\rangle : G \text{ contains an independent set of size } k\}$$

Clearly IND-SET $\in \mathcal{NP}$. We show that CLIQUE $\leq_P$ IND-SET and deduce that IND-SET $\in \mathcal{NPC}$

CLIQUE $\leq_P$ IND-SET

Proof:

Definition 37

The complement of a graph $G = (V, E)$ is a graph $G^c = (V, E^c)$, where $E^c = \{(v_1, v_2) : v_1, v_2 \in V \text{ and } (v_1, v_2) \notin E\}$.

The reduction $f(G, k)$ from CLIQUE to IND-SET simply computes the complement of the graph and outputs (G^c, k).

f satisfies:

- U is a clique in G $\iff$ U is a independent set in G^c.
- computable in polynomial time!

♣

Remark 38

Same reduction shows that IND-SET $\leq_P$ CLIQUE

Section 6

Satisfiability

Boolean Variables

- A Boolean variable assumes values
  - TRUE (written 1), and FALSE (written 0).
- Boolean operations:
  - and: $\wedge$
  - or: $\vee$
  - not: $\neg$
- Examples:

$$\begin{aligned} 0 \wedge 1 &= 0 \\ 0 \vee 1 &= 1 \\ \overline{0} &= 1 \end{aligned}$$

Boolean Formulas and SAT

A Boolean formula is an expression involving Boolean variables and operations.

$$\phi = (\overline{x} \wedge y) \vee (x \wedge \overline{z})$$

Definition 39 (satisfiable formula)

A formula is satisfiable, if some Boolean assignment to its variables, makes the formula evaluate to 1.

The formula $\phi = (\overline{x} \wedge y) \vee (x \wedge \overline{z})$ is satisfiable by the assignment

$$\begin{aligned} x &= 0 \\ y &= 1 \\ z &= 0 \end{aligned}$$

The language of satisfied formulas:

$$\text{SAT} = \{\langle\phi\rangle : \ \phi \text{ is a satisfiable Boolean formula}\}$$

SAT $\in \mathcal{NPC}$

$$\text{SAT} = \{\langle\phi\rangle : \ \phi \text{ is satisfiable Boolean formula}\}$$

Theorem 40 (Cook-Levin (early 70s))

SAT $\in \mathcal{NPC}$.

- The “most important" $\mathcal{NP}$-complete language.
- It is easy to see that SAT $\in \mathcal{NP}$
- For the proof of other part wait for next week . . .

The Language 3SAT

It is useful to consider a special version of SAT

- A literal is a variable or negated variable: x or $\overline{x}$.
- A clause is several literals joined by $\vee$s: $(x_1 \vee \overline{x_2} \vee \overline{x_3})$
- A Boolean formula is in conjunctive normal form (CNF) if it consists of clauses, connected with $\wedge$s.

  For example: $(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6})$

Definition 41

A Boolean formula is in k-CNF form, if it is a CNF formula, and all clauses have k literals.

- Example of 3CNF: $(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6} \vee x_4)$

The language of satisfied 3CNF formulas:

$$3SAT = \{\langle \phi \rangle : \ \phi \text{ is satisfiable 3CNF formula}\}$$

3SAT $\in \mathcal{NPC}$

- Clearly 3SAT $\leq_P$ SAT.

  and 3SAT $\in \mathcal{NP}$.
- We show next class that SAT $\leq_P$ 3SAT, yielding that 3SAT $\in \mathcal{NPC}$.
- Since 3SAT has more structure than SAT, it is simpler to reduce it to other languages.
- In the following we prove that several $\mathcal{NP}$ languages are $\mathcal{NPC}$, by showing a reduction from 3SAT to them.

1SAT, 2SAT $\in \mathcal{P}$

Question 42

Is 1SAT $\in \mathcal{P}$?

Question 43

Is 2SAT $\in \mathcal{P}$?

An algorithm for $2CNF$

The following slides on the algorithm not done in class. Not deleted in case you are interested in reading them, but not required. What you need to know is the fact that $2CNF$ is in P.

A Graph characterization for $2CNF$

Definition 44

For $2CNF$ formula ϕ with variables $x_1, \ldots, x_k$, define a directed graph $G(\phi) = (V, E)$ as

- $V = \{x_1, \overline{x_1}, \ldots, x_k, \overline{x_k}\}$
- $E = \{(\overline{\ell}, h), (\overline{h}, \ell) : (\ell \vee h) \in \phi\}$ (taking $\overline{\overline{x}} = x$).

Claim 45

$G(\phi)$ contains path from ℓ to $h \implies$ in any satisfying assignment of ϕ with $\ell = 1$, it holds that $h = 1$.

Proof?

Claim 46

$G(\phi)$ contains path from ℓ to $h \implies G(\phi)$ contains path from $\overline{h}$ to $\overline{\ell}$

Proof?

A Graph characterization for $2CNF$, cont

Definition 47

A variable x in $2CNF$ ϕ is bad, if $\exists$ paths in $G(\phi)$ from x to $\overline{x}$ and from $\overline{x}$ to x.

Claim 48

A $2CNF$ formula is satisfiable iff it contains no bad variables.

- ϕ has bad variable $\implies$ ϕ is unsatisfiable. Proof? by Claim 45
- ϕ has no bad variables $\implies$ ϕ is satisfiable. Proof?

A Graph characterization for $2CNF$, cont

Definition 49

A variable x in $2CNF$ ϕ is bad, if $\exists$ paths in $G(\phi)$ from x to $\overline{x}$ and from $\overline{x}$ to x.

Claim 50

A $2CNF$ formula is satisfiable iff it contains no bad variables.

Algorithm 51

While ϕ has unassigned variables:

1. Pick unassigned literal $\ell \in V$ that has no path from ℓ to $\overline{\ell}$
2. Assign 1 to all literals reachable from ℓ, and 0 to their negations.
3. Recompute graph for "reduced formula"

Can there be conflicts (i.e., x and $\overline{x}$ assigned the same value)?

- Same iteration? Assume $\ell \rightsquigarrow h \rightsquigarrow \overline{h}$. $\implies$ $h \rightsquigarrow \overline{h} \rightsquigarrow \overline{\ell}$ $\implies$ $\ell \rightsquigarrow \overline{\ell}$.
- Different iterations? Removing edges cannot add bad variables.

Final assignment satisfies ϕ. Assume h is set to 1, then all clauses it participates in are satisfied.

Poly-time algorithm for 2SAT

Algorithm 52 (TwoSatSolver)

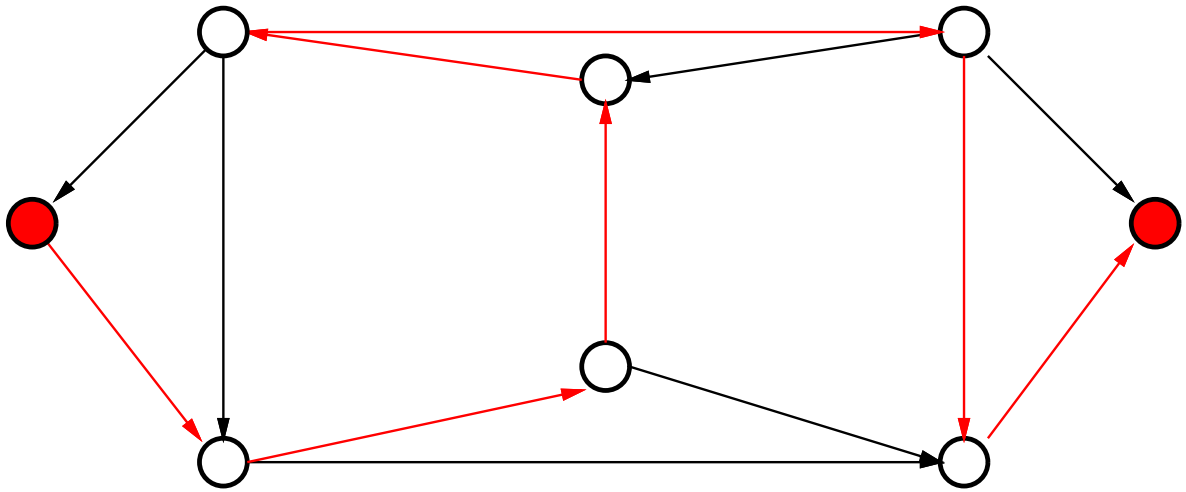
Input: *2CNF* ϕ
Return `TRUE` if there exist no bad variables in ϕ:

- Efficiency?
- Correctness?

Section 7

Hamiltonian Paths and Cycles

Reminder – Hamiltonian Path



A Hamiltonian path in a directed G visits each node exactly once.

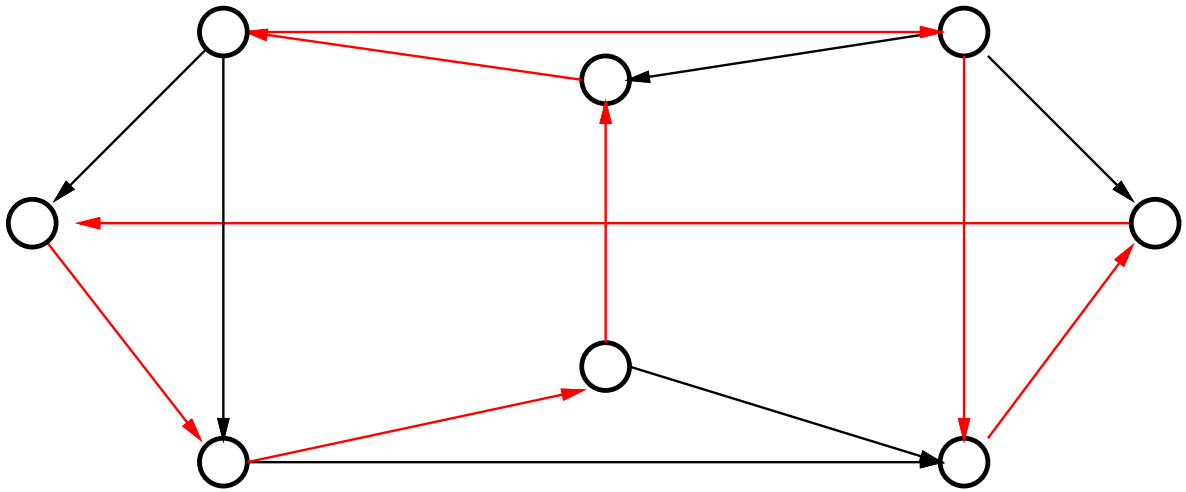
$$\text{HAMPATH} = \{\langle G, s, t\rangle : G \text{ has Hamiltonian path from } s \text{ to } t\}$$

Theorem 53

HAMPATH $\in \mathcal{NPC}$.

Not today . . .

Hamiltonian Cycle



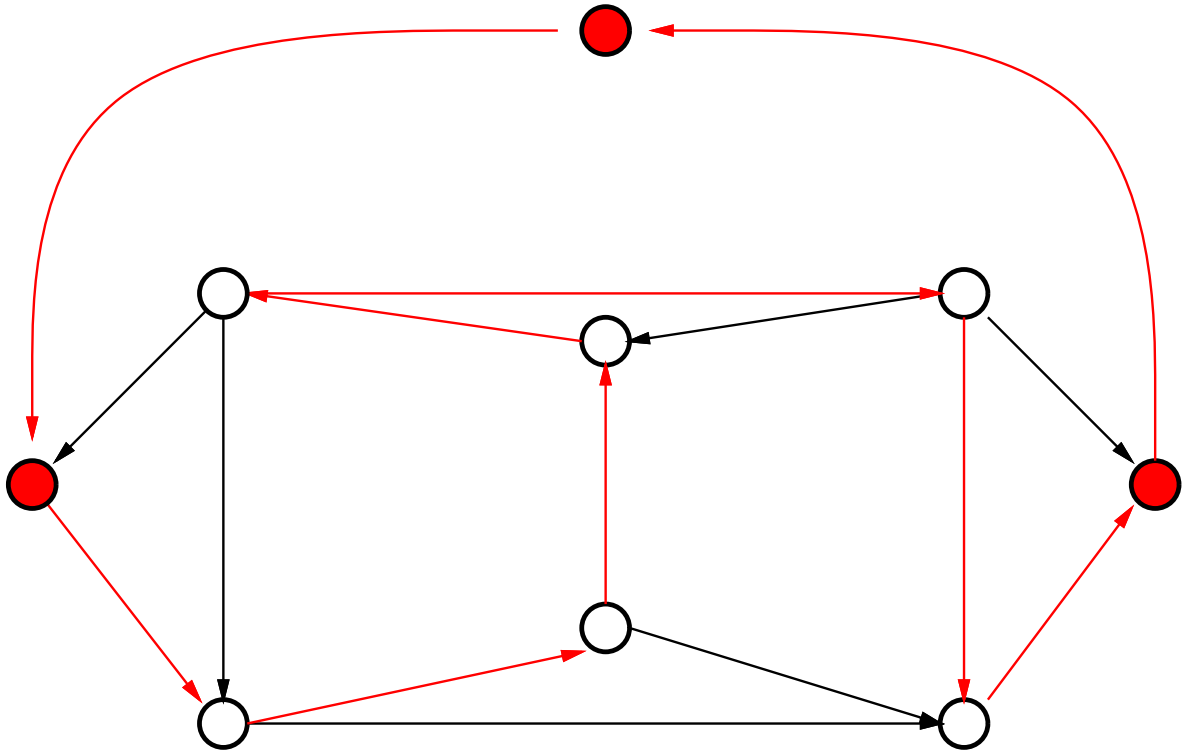
A Hamiltonian Cycle in a graph is an Hamiltonian path that ends up where it starts (i.e., $s = t$).

$$\text{HAMCYCLE} = \{\langle G \rangle : G \text{ has Hamiltonian cycle}\}$$

Clearly HAMCYCLE $\in \mathcal{NP}$. We will show that HAMPATH $\leq_P$ HAMCYCLE, and deduce that HAMCYCLE $\in \mathcal{NPC}$

HAMPATH $\leq_P$ HAMCYCLE.

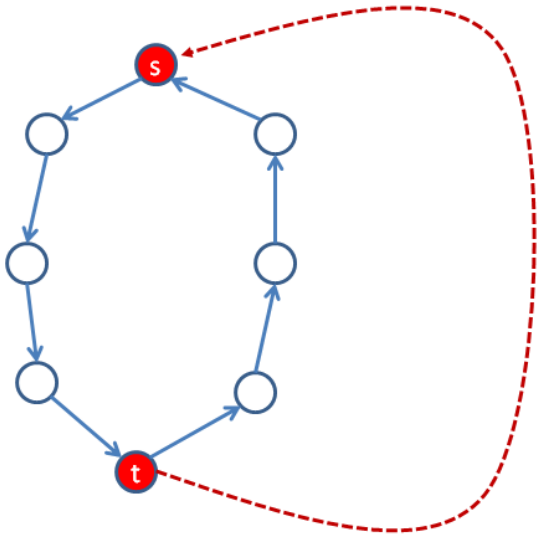
Proof's idea:



Hey, is the new vertex really needed? Why not just add an edge from t to s?

HAMPATH $\leq_P$ HAMCYCLE.

Why the new vertex really needed? Why not just add an edge from t to s?



HAMCYCLE $\leq_P$ HAMPATH.

Claim 54

HAMCYCLE $\leq_P$ HAMPATH.

Left as an easy (recommended) exercise.

Undirected Hamiltonian Circuit

$$\text{UHAMCYCLE} = \{\langle G \rangle : \ G \text{ is undirected } \& \text{ has Hamiltonian cycle}\}$$

where Hamiltonian cycle/path in an undirect graph, is defined analogously to the direct case.

Clearly $\text{UHAMCYCLE} \in \mathcal{NP}$.

It is not hard to show (see Sipser 7.55, for a similar proof) that $\text{HAMCYCLE} \leq_P \text{UHAMCYCLE}$, and deduce that $\text{UHAMCYCLE} \in \mathcal{NPC}$

Chains of Reductions: NPC Problems