Exercise 2 - Computational Models - Spring 2016

1. We showed in class that Regular languages are closed under union. Are they closed under infinite union? Prove your answer formally.

2. (a) For each of the following, write a regular expression for $h(L)$:

   i. $L = L(01(01)^*11)$ and the homomorphism $h : \{0, 1\} \to \{a, b, c\}^*$, s.t. $h(0) = abb,\ h(1) = c$

   ii. $L = L((00 \cup 1)^*)$ and the homomorphism $h : \{0, 1\} \to \{a, b\}^*$ s.t. $h(0) = b$ and $h(1) = \varepsilon$

   iii. $L = \{abab, baba\}$ and the homomorphism $h : \{a, b\} \to \{0, 1\}^*$ s.t. $h(a) = 01$ and $h(b) = 11$

   (b) For each of the following, write a regular expression for $h^{-1}(L)$:

   i. $L = L(01(01)^*11)$ and the homomorphism $h : \{a, b, c\} \to \{0, 1\}^*$, s.t. $h(a) = 0,\ h(b) = 01$ and $h(c) = 1$

   ii. $L = L((00 \cup 1)^*)$ and the homomorphism $h : \{a, b\} \to \{0, 1\}^*$ s.t. $h(a) = 01$ and $h(b) = 10$

   iii. $L = \{abab, baba\}$ and the homomorphism $h : \{0, 1\} \to \{a, b\}^*$ s.t. $h(0) = ab$ and $h(1) = \varepsilon$

3. Prove or disprove:

   (a) If L_1 and L_2 are non regular context free languages then $L_1 \cup L_2$ is not regular.

   (b) For any homomorphism h, If L is not regular then $h(L)$ is not regular

4. (a) Let M be a DFA with n states. Prove that $L(M)$ is infinite if and only if $\exists w \in L(M)$ s.t. $n < |w| \leq 2n$.

   (b) Describe an algorithm that given a DFA M, decides if $L(M)$ is infinite.

   (c) Describe an algorithm that given a DFA M, decides if $|L(M)| = 9,122,009$.

5. For each of the following languages, present a formal definition of a CFG (no need for a correctness proof, but do provide an explanation).

   (a) $\{w \mid \#_0(w) > \#_1(w)\}$ over $\Sigma = \{0, 1\}$
   (b) $\{x\$y \mid x^R$ is a substring of y and $x, y \in \{0, 1\}^*\}$ over $\Sigma = \{0, 1, \$\}$

6. Use the pumping lemma to show that the following languages are not context-free.

   (a) $L_1 = \{x\$y \mid x, y \in \{0, 1\}^*$ and x is a substring of $y\}$ over $\Sigma = \{0, 1, \$\}$
   (b) $L_2 = \{0^n\$0^{2n}\$0^{3n} \mid n \geq 0\}$ over $\Sigma = \{0, \$\}$

7. For languages A and B, let the *shuffle* of A and B be the language

   $\{w \mid w = a_1b_1 \cdots a_kb_k$, where $a_1 \cdots a_k \in A$ and $b_1 \cdots b_k \in B$, each $a_i, b_i \in \Sigma^*\}$

   Prove or disprove:

   (a) The regular languages are closed under this operation.
   (b) The context free languages are closed under this operation.