Computational Models, Spring 2016 Exrecise #4

Turing Machines, $\mathcal{R}$, $\mathcal{RE}$ co-$\mathcal{RE}$

1. Let $A = \{Q_A, \Sigma, \delta_A, q_{0A}, F_A\}$ be a DFA Construct *formally* a TM that accepts $L(A)$
2. In this question we are concerned with $<M>$, the encoding of M according to the convention we seen in class.
   (a) Let $M = \{Q, \Sigma, \Gamma, \delta, q_1, q_2, q_3\}$ be a TM such that: $Q = \{q_1, q_2, q_3\}$ $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, \sqcup\}$ and δ is defined as follows: $\delta(q_1, 0) = (q_1, 0, R)$; $\delta(q_1, 1) = (q_3, 1, L)$; $\delta(q_1, \sqcup) = (q_2, \sqcup, L)$
      i. What is the language accepted by the TM? (no need to prove your answer).
      ii. What us $<M>$?
   (b) Write an algorithm (in pseudocode) that on input w, checks that w encodes a valid TM $<M>$. E.g., you need to validate that the structure is correct, that $\delta(q_2, a)$ is undefined for every symbol a, etc. (recall that q_2 is by convention the accepting state).
3. Give a verbal description of a Turing Machine that accepts the following languages. There is no need to give a formal definition but you should give a detailed explanation of each step.
   (a) $L = \{w \in \Sigma^* \mid \#_a(w) = \#_b(w) = \#_c(w)\}$ above $\Sigma = \{a, b, c\}$
   (b) $L = \{\{0\}^{n^2} \mid n \in N \text{ and } n > 0\}$ above $\Sigma = \{0, 1\}$
   (c) $L = \{a^i b^j c^k \mid i \cdot j = k \text{ and } i, j, k \geq 0\}$ above $\Sigma = \{a, b, c\}$
4. Let us define a generalization of Turing Machines to include a *finite memory* of size n. We denote such a Turing Machine formally as:

   $$M_{\text{mem}} = (Q, \Sigma, \gamma, \delta_{\text{mem}}, n, q_0, qa, q_r),$$

   where all the definitions are identical to the Turing Machine defined in class except that there is the *finite* memory size n and the transition function δ_{mem}. At each step, the transition depends on the current state, the input on the tape and all the memory. The transition to the next step can update the entire memory. Formally: $\delta_{\text{mem}} : Q \times \Gamma \times \Gamma^n \to Q \times \Gamma \times \Gamma^n \times \{L, R\}$.

   Given a finite memory Turing Machine M_{mem}, define *formally* a (standard) Turing Machine M such that $L(M) = L(M_{\text{mem}})$. Namely, both Turing Machines accept the same language. Explain your answer.

   Bonus: Show more than one formal construction.
   Bonus: Give a formal proof.
5. Prove or disprove:
   (a) $\mathcal{R}$ is closed under complementation.
   (b) $\mathcal{RE}$ is closed under complementation.
   (c) $\mathcal{RE}$ is closed under intersection.
   (d) co-$\mathcal{RE}$ is closed under intersection.
   (e) $\mathcal{RE}$ is closed under Kleene star.
6. Let $L_1, L_2 \in RE \setminus R$ can the following occur? If your answer is "yes" give a concrete example. If your answer is "no" give a sketch of a proof.
   (a) $L_1 \cap L_2 \in R$
   (b) $L_1 \cup L_2 \in R$
   (c) $L_1 \cap L_2 \in R$ and $L_1 \cup L_2 \in R$
7. Prove or contradict: R is closed under infinite (countable) union. Namely, given $L_1, L_2 \ldots$ such that $L_i \in R$ then $\bigcup_{i=1}^{\infty} L_i \in R$.