Computational Models - Exercise #5, Spring 2015/16

Due: May 30

1. Let A, B and C languages over Σ. Prove/disprove:
   (a) If $A \leq_m B$ and $B \leq_m C$ then $A \leq_m C$.
   (b) If $A \leq_m B$ and $B \leq_m A$ then $A = B$.
   (c) If $A \subseteq B$ then $A \leq_m B$.
   (d) For every A and B, either $A \leq_m B$ or $B \leq_m A$.
   (e) If A is context-free then $A \leq_m H_{TM}$.

2. For the following decision problems, determine whether they belong to R, $\mathsf{RE} \setminus \mathsf{R}$, $\mathsf{coRE} \setminus \mathsf{R}$ or outside of $\mathsf{RE} \cup \mathsf{coRE}$:
   (a) $L_1 = \{\langle M \rangle \mid M \text{ is a TM and } |L(M)| \geq 2016\}$.
   (b) $L_2 = \{\langle M, x, y \rangle \mid M \text{ is a TM that halts on exactly one of the inputs}\}$.
   (c) $L_3 = \{\langle M \rangle \mid |\langle M \rangle| \leq 2016 \text{ and } M \text{ is a TM that accepts } \varepsilon\}$.
   (d) $L_4 = \{\langle M \rangle \mid M \text{ is a TM and for every } x, M\text{'s run on } x \text{ never reaches position } |x|+20\}$.
   (e) $L_5 = \{\langle M \rangle \mid M \text{ is a TM and } A_{TM} \leq_m L(M)\}$.

3. Prove/disprove that the following languages are decidable (in R):
   (a) $L_6 = \{\langle M \rangle \mid \text{There exists an input that the TM } M \text{ accepts in less than 50 steps}\}$.
   (b) $L_7 = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \cup H_{TM} \in \mathsf{RE}\}$.
   (c) $L_8 = \{\langle D, R \rangle \mid D \text{ is a DFA, } R \text{ is a regular expression and } L(D) = L(R)\}$.
   (d) $L_9 = \{\langle M \rangle \mid M \text{ is a TM that never rejects}\}$.

4. Let $\mathcal{C}$ be the set of context-free languages. Prove/disprove:
   (a) For every $A \in \mathcal{C}$, $L_A = \{\langle G \rangle \mid \text{G is a CFG and } L(G) = A\} \in \mathsf{R}$.
   (b) Let $C \subsetneq \mathcal{C}$ with $|C| > 1$. Then, $L_C = \{\langle G \rangle \mid \text{G is a CFG and } L(G) \in C\} \notin \mathsf{R}$.

5. Assume that we work with one-tape TM, with $\Sigma = \{0, 1\}$ and $\Gamma = \{0, 1, \llcorner\!\lrcorner\}$. For every $n \in \mathbb{N}$, define S_n to be the set of Turing machine with n states that halt on ε. Let $B(n)$ be the maximum number of steps taken by some TM from S_n until it halts. Note that by our definition of S_n, $B(n)$ is a total function from $\mathbb{N}$ to $\mathbb{N}$.
   (a) Prove that the function B is not computable. That is, that there exists no TM that on input n (for every n) writes $B(n)$ on its tape and halts.
   Hint: Assume towards contradiction that B is computable and show how to decide $H_{TM,\varepsilon}$.

(b) Let $f : \mathbb{N} \to \Sigma^\star$ be such that $f(n)$ returns the n-th word in $\overline{H_{TM,\varepsilon}}$. Prove/disprove: f is computable.

6. We say that a language L has a *verifier* if there exists a deterministic TM V that always halts, and for every $x \in \Sigma^\star$:

   - If $x \in L$ then there exists $c \in \Sigma^\star$ such that V accepts (x, c).
   - If $x \notin L$ then for every $c \in \Sigma^\star$, V rejects (x, c).

   Prove: L has a verifier if and only if $L \in \mathsf{RE}$.

7. Let F be the set of computable total functions and let $\emptyset \subsetneq S \subseteq F$. Define

$$L_S = \{\langle M \rangle \mid M \text{ is a TM that computes a function and } f_M \in S\},$$

   where f_M is the function that M computes. Prove that for every such S, $L_S \notin \mathsf{R}$.

8. Let $L_1, L_2, \ldots$ be an enumeration of R and define $A_i = \{\langle M \rangle \mid L(M) = L_i\}$.

   Let L be a language in RE such that $L \subseteq \{\langle M \rangle \mid M \text{ is a TM that always halts}\}$. Prove that there exists an i for which $L \cap A_i = \emptyset$. Hint: Build a TM that returns some (which?) element of L and apply a diagonalization argument.