Computational Models - Exercise #6, Spring 2015/16

Due: June 10

Note: We denote $A \leq_p B$ if there exists a polynomial-time reduction from A to B. We denote $A \leq_m B$ if there exists a mapping-reduction from A to B. A language L is nontrivial if $L \neq \emptyset$ and $L \neq \Sigma^\star$.

1. Determine whether the following claims are true, false or equivalent to an open problem:
   (a) For every nontrivial $L_1, L_2 \in \mathsf{P}$, if $L_1 \leq_m L_2$ then $L_1 \leq_p L_2$.
   (b) For every nontrivial $L_1, L_2 \in \mathsf{NP}$, if $L_1 \leq_m L_2$ then $L_1 \leq_p L_2$.
   (c) There exists a language in RE that is complete w.r.t. polynomial-time reductions.
   (d) $L = \{xx \mid x \in \{0,1\}^\star\}$ is NP-complete.
   (e) If there exists a deterministic TM that decides SAT in time $n^{O(\log n)}$ then every $L \in \mathsf{NP}$ is decidable by a deterministic TM in time $n^{O(\log n)}$.

2. Assuming $\mathsf{P} \neq \mathsf{NP}$, determine whether the following languages are in P or not.
   (a) $L_1 = \{\langle\varphi\rangle \mid \varphi$ is a satisfiable CNF and has at most 2016 clauses$\}$.
   (b) $L_2 = \{\langle\varphi\rangle \mid \varphi$ is a satisfiable CNF and has at least 2016 clauses$\}$.
   (c) $L_3 = \{\langle G, k\rangle \mid G$ is a graph where every vertex has degree 2 and G has a vertex cover of size $k\}$.
   (d) $L_4 = \{\langle G, e\rangle \mid G$ is a graph that has a Hamiltonian cycle passing through the edge $e\}$.

3. Prove:
   (a) If $L \in \mathsf{NP}$ then $L^\star \in \mathsf{NP}$.
   (b) If $L \in \mathsf{P}$ then $L^\star \in \mathsf{P}$.

4. We say that a polynomial reduction f is a *shrinking reduction* if there exists n_0 such that for every $x \in \Sigma^\star$ such that $|x| \geq n_0$, $|f(x)| \leq |x| - 1$. Assuming $\mathsf{P} \neq \mathsf{NP}$, prove/disprove:
   (a) For every two nontrivial languages $A, B \in \mathsf{P}$ there exists a shrinking reduction from A to B.
   (b) For every two nontrivial languages $A, B \in \mathsf{NPC}$ there exists a shrinking reduction from A to B.

5. We say that two languages A and B are *polynomially equivalent* if $A \leq_p B$ and $B \leq_p A$. Prove/disprove:

(a) VC[1], CLIQUE and IS are polynomially equivalent[2].

(b) H_{TM} and $\overline{H_{TM}}$ are polynomially equivalent.

(c) Assuming $\mathsf{P} \neq \mathsf{NP}$, every two nontrivial languages in NP are polynomially equivalent.

6. We say that $L \in \oplus\mathsf{P}$ if there exists a non-deterministic polynomial TM M such that:

- If $x \in L$ the number of accepting paths of M on x is odd.
- If $x \notin L$ the number of accepting paths of M on x is even.

Prove/disprove:

(a) If $L \in \oplus\mathsf{P}$ then there exists a deterministic TM M such that $L(M) = L$, M always halts and uses polynomial number of cells.

(b) If $L_1, L_2 \in \oplus\mathsf{P}$ then $L_1 \cap L_2 \in \oplus\mathsf{P}$.

7. Assuming $\mathsf{P} \neq \mathsf{NP}$, determine whether the following languages are in P or not. S_i for every i and T are positive integers. $[n]$ denotes the set $\{1, \ldots, n\}$.

(a) $L_1 = \left\{(S_1, \ldots, S_n, T, m) \mid \exists I \subseteq [n]. |I| = m \wedge \sum_{i \in I} S_i = T\right\}$.

(b) $L_2 = \left\{(S_1, \ldots, S_n, T) \mid \exists I \subseteq [n]. \left|\sum_{i \in I} S_i - T\right| \leq 10\right\}$.

(c) $L_3 = \left\{(S_1, \ldots, S_n, T) \mid \exists I \subseteq [n]. \forall i \in I. S_i \geq \frac{T}{2016} \wedge \sum_{i \in I} S_i = T\right\}$.

[1] A vertex cover of an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ such that if $\{u, v\} \in E$ then $u \in V'$ or $v \in V'$ (or both). The language VC is the language of all $(\langle G \rangle, k)$ such that G has a vertex cover of size k.

[2] You *cannot* use the fact that all three problems are in NPC, but you *can* state and prove the transitivity of polynomial reductions.