Solution sketch 2 - Computational Models - Spring 2016

1. No, consider $L_i = \{a^i b^i\}$ which is regular for every i (finite language). Their union is $\{a^n b^n \mid n \in \mathbb{N}\}$ which is not regular.

2. (a) i. $abbc(abbc)^*cc$
   ii. $(bb)^*$
   iii. $(01110111 \cup 11011101)$

   (b) i. $(ac \cup b)(ac \cup b)^*cc$
   ii. $(ba)^*$
   iii. $1^*01^*01^*$

3. (a) False. Let $L_1 = \{0^n 1^m \mid n, m \in \mathbb{N}, n \leq m\}$ and $L_2 = \{0^n 1^m \mid n, m \in \mathbb{N}, n \geq m\}$. Both are non regular context free languages and $L_1 \cup L_2 = 0^*1^*$ is regular.

   (b) False. Consider $L = \{0^n 1^n \mid n \in \mathbb{N}\}$ and the homomorphism $h(0) = a,\ h(1) = \varepsilon$. L is not regular but $h(L) = a^*$ is regular.

4. (a) If $\exists w$ s.t. $|w| > n$ then as we learned in the pumping lemma, this word can be pumped infinitely and therefore $L(M)$ is infinite. if $L(M)$ is infinite, then there is a word $w \in L(M)$ such that $|w| > n$. The run of this word in M contains a cycle. We remove all cycles from the run and remember one simple cycle c, $|c| \leq n$. The run without the cycles give a word $w' \in L(M),\ |w'| < n$. We start pumping w' with the cycle c and we will eventually get a word in $L(M)$ in the proper length.

   (b) Using 4a, given a DFA M, we can run in M all the words w, such that $n < |w| \leq 2n$. If one of the words is accepted, then $L(M)$ is infinite, otherwise - finite.

   (c) First we check if $L(M)$ is infinite. If it is, we return $false$. otherwise, we run in M all words of length at most n and count how many are accepted. we return $true$ iff the count is 9,122,009.

5. (a) $G = (\{S, A\}, \{0, 1\}, R, S)$ where R has the rules

$$\begin{aligned} S &\rightarrow A0A \\ A &\rightarrow AA \mid 0A1 \mid 1A0 \mid 0 \mid \varepsilon \end{aligned}$$

A generates all strings with at least as many 0 as 1, and S force an extra 0.

(b) $G = (\{S, A, B\}, \{0, 1, \$\}, R, S)$ where R has the rules

$$\begin{aligned} S &\rightarrow AB \\ A &\rightarrow 0A0 \mid 1A1 \mid \$B \\ B &\rightarrow 0B \mid 1B \mid \varepsilon \end{aligned}$$

A generates strings of the form $w^R\$wB$ and B generates all strings in $\{0, 1\}^*$

6. (a) For any p choose $w = 0^p1^p\$0^p1^p$. $|w| \geq p$ and $w \in L_1$. Let $w = uvxyz$ s.t. $|vy| > 0$ and $|vxy| \leq p$.

   - If y or v contains \$ $uv^0xy^0z = uxz \notin L_1$ since there is no \$ in it.
   - Otherwise, 3 possible cases:
     i. vxy is in the first part of w (before the \$), $uv^2xy^2z \notin L_1$ because the first part is longer than the second part.
     ii. vxy is in the second part of w (after the \$), $uv^0xy^0z \notin L_1$ because the first part is longer than the second part.
     iii. If vxy is in the middle of w, i.e. $vxy = 1^i\$0^j$
       – If $j > 0$, then $uv^0xy^0z \notin L_1$ (less 0 in the second part)
       – If $i > 0$, then $uv^2xy^2z \notin L_1$ (more 1 in the first part)

(b) For any p choose $w = 0^p\$0^{2p}\$0^{3p}$. $|w| \geq p$ and $w \in L_2$. Let $w = uvxyz$ s.t. $|vy| > 0$ and $|vxy| \leq p$. If y or v contains \$ $uv^2xy^2z \notin L_2$ since it contains more than two \$ in it. If we divide w into three segments by \$: 0^p, 0^{2p}, and 0^{3p}. At least one of the segments is not contained within either v or y. Hence $uv^2xy^2z \notin L_2$ because $1:2:3$ length ratio of the segments is not maintained.

7. (a) The regular languages are closed under this operation. Let $M_A = (Q_A, \Sigma, \delta_A, q_0^A, F_A)$ and $M_B = (Q_B, \Sigma, \delta_b, q_0^B, F_B)$ be two DFAs that recognize A and B, respectively. We build an NFA $N = (Q, \Sigma, \delta, q_0, F)$ that recognizes the $shuffle$ of A and B as follow:

   - $Q = Q_A \times Q_B$

- $q_0 = (q_0^A, q_0^B)$
- $\forall \sigma \in \Sigma, q_a \in A, q_b \in B \;\; \delta((q_a, q_b), \sigma) = \{(\delta_A(q_a, \sigma), q_b)\} \cup \{(q_b, \delta_B(q_b, \sigma))\}$
- $F = F_A \times F_B$

You should prove that $L(N)$ is equal to the $shuffle$ of A and B.

(b) The context free languages are not closed under this operation. Define A an B as follows

$$A = \{w \in \{0, 1\}^* \mid \#_0(w) = \#_1(w)\}$$

$$A = \{w \in \{a, b\}^* \mid \#_a(w) = \#_b(w)\}$$

A and B are both context free languages. The shuffle of A and B is the language

$$S = \{w \in \{0, 1, a, b\}^* \mid \#_0(w) = \#_1(w) \text{ and } \#_a(w) = \#_b(w)\}$$

Assume that S is context-free. For any p choose $w = 0^p a^p 1^p b^p$. Thus $w \in S$ and $|w| \geq p$. Let $w = uvxyz$ s.t. $|vy| > 0$ and $|vxy| \leq p$. Since $|vxy| \leq p$, it cannot contain both zeros and ones or both a's and b's. Hence uv^2xy^2z will result in a string with either an unbalanced number of zeros and ones or an unbalanced number of a's and b's or both. Thus, $uv^2xy^2z \notin S$, so S is not context-free.